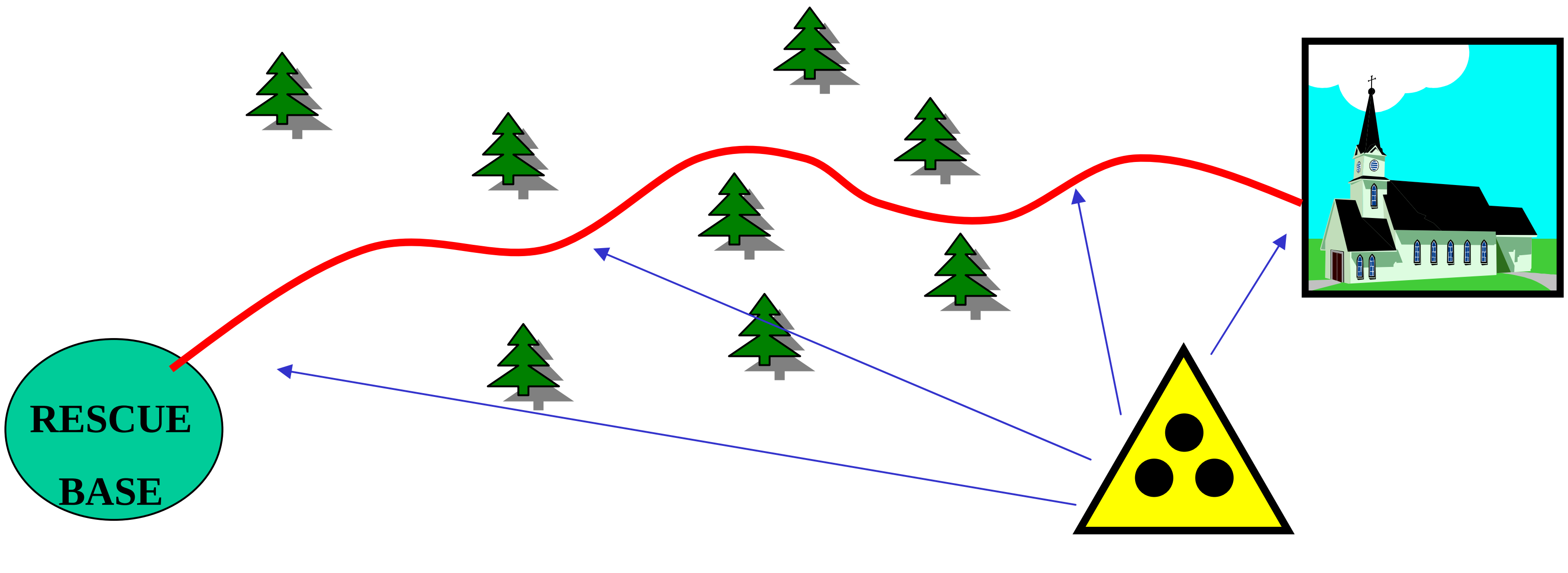
LINE INTEGRAL IN A SCALAR FIELD

MOTIVATION

A rescue team follows a path in a danger area where for each position the degree of radiation is defined. Compute the total amount of radiation gathered by the rescue team along the path.

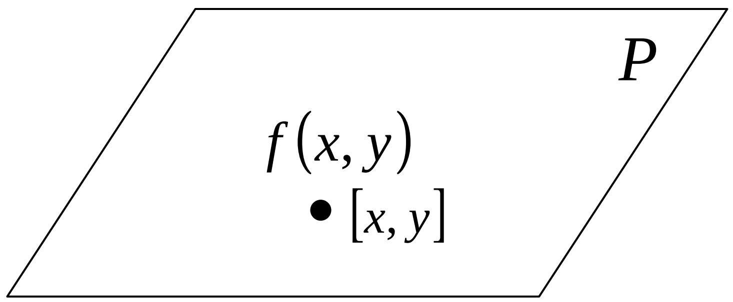


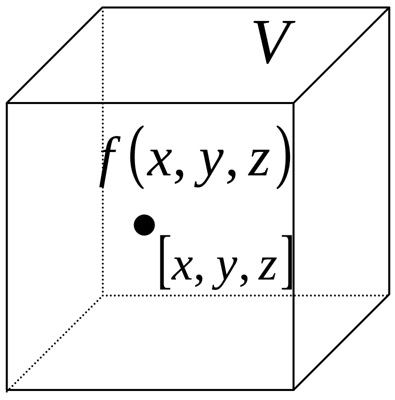
SCALAR FIELD

Let P be a planar contiguous area with a function $f(x,y)$ defined on it. We call $f(x,y)$ a scalar field defined over M

Similarly, for a 3D area V, we define a scalar field as a function $f(x,y,z)$ with the domain V.

temperature, radiation, moisture, mass, ...





Let a scalar field $f(x,y)$ be defined in a planar area M.

Let L be a regular curve defined by the parametric equations

$$x=\varphi(t) \quad y=\psi(t), \; t\in[a,b]$$

Define a partition of T:

$$a = a_0 < a_1 < \cdots < a_n$$

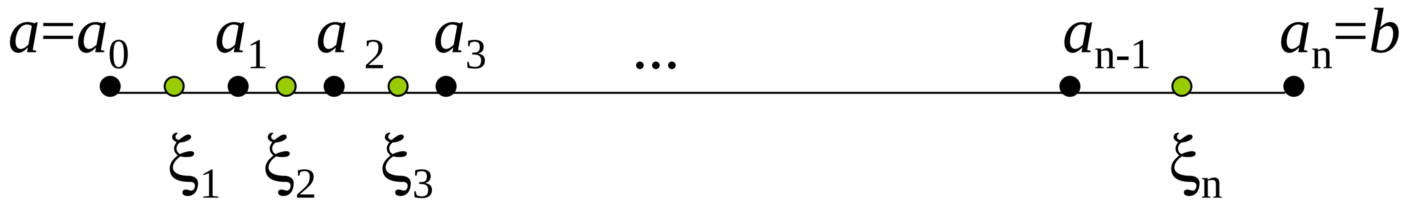
Put $L_0 = [\varphi(a_0), \psi(a_0)], L_1 = [\varphi(a_1), \psi(a_1)], \ldots, L_n = [\varphi(a_n), \psi(a_n)]$

Choose points $\xi_1, \xi_2, \ldots, \xi_n$ such that

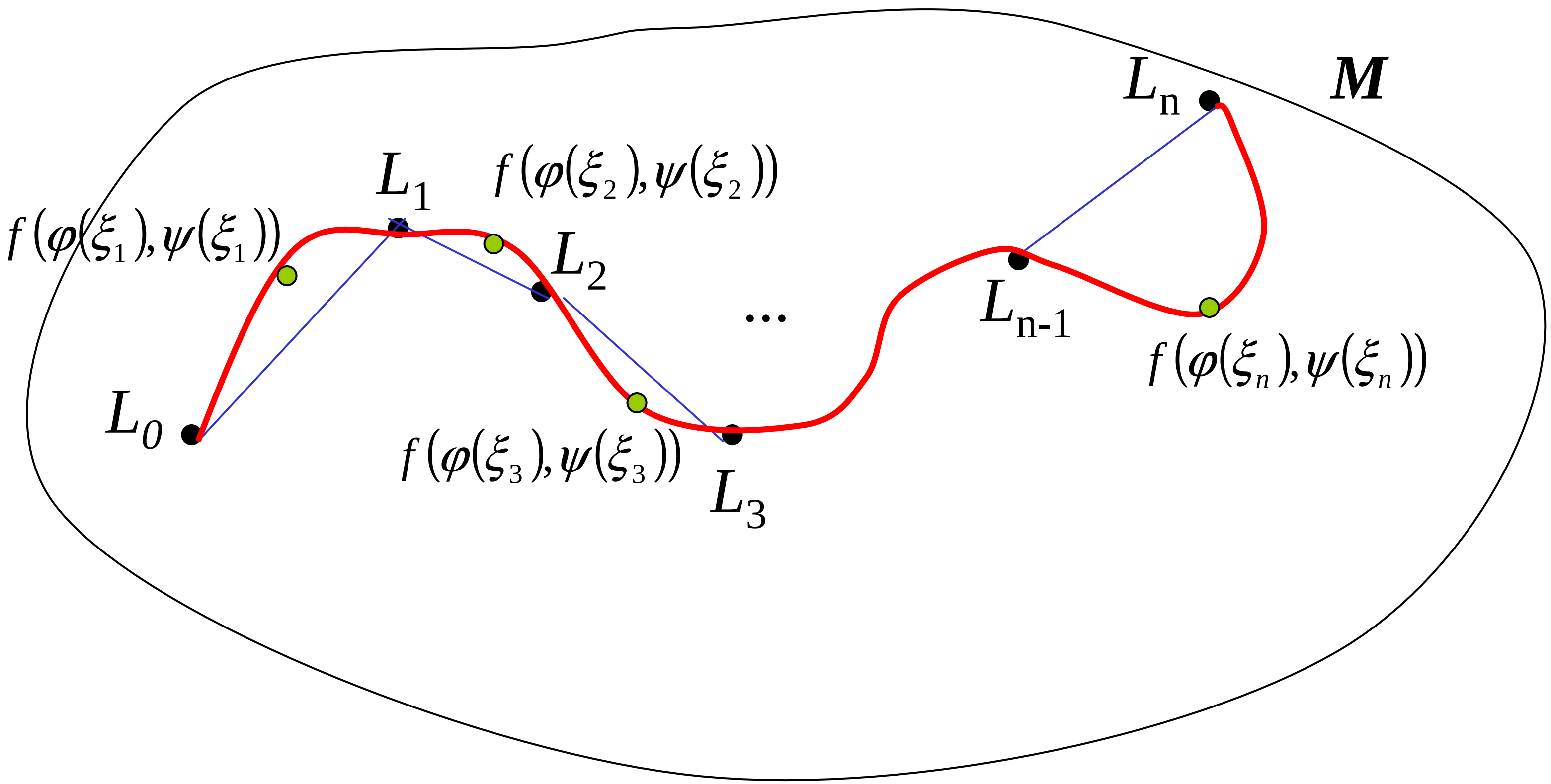
$$a_0 < \xi_1 < a_1 < \xi_2 \cdots < \xi_n < a_n$$

Put $N(a_0, a_1, \ldots, a_n) = \max_{0<i\le n}\left\{\overline{L_i L_{i-1}}\right\}$

$N(a_0, a_1, \ldots, a_n)$ is called the norm of the partition.



a=a0
a1
a 2
a3
...
an-1
an=b
ξ1
ξ2
ξ3
ξn



Ln
M
L1
f(φ(ξ2),ψ(ξ2))
f(φ(ξ1),ψ(ξ1))
L2
...
Ln-1
f(φ(ξn),ψ(ξn))
L0
f(φ(ξ3),ψ(ξ3))
L3

We can define the following integral sum

$$\mathcal{L} = \overline{L_0 L_1}\, f\left(\varphi(\xi_1), \psi(\xi_1)\right) + \overline{L_1 L_2}\, f\left(\varphi(\xi_2), \psi(\xi_2)\right) + \cdots + \overline{L_{n-1} L_n}\, f\left(\varphi(\xi_n), \psi(\xi_n)\right)$$

If, for $N(a_0, a_1, \ldots, a_n) \to 0$, we have $\mathcal{L} \to \mathcal{L}_0$ for some finite $\mathcal{L}_0$,

we say that $\mathcal{L}_0$ is the ***line*** (or ***curvilinear***) ***integral*** of the scalar field $f(x,y)$ over L (line integral of the first type).

Formally, we write

$$\mathcal{L}_0 = \int_L f(x, y)\, ds$$

$\mathscr{L}_0 = \int_L f(x, y)\,ds$ is independent of the way the partition of

$[a,b]$ is constructed and the numbers $\xi_1, \xi_2, \ldots, \xi_n$ chosen.

If $f(x,y)$ is continuous on a regular curve L given by the

parametric equations $x=\varphi(t)$ $y=\psi(t)$, $t\in[a,b]$ then the following

formula can be used

$$\int_L f(x, y)\,ds = \int_a^b f(\varphi(t), \psi(t))\sqrt{\varphi'^2(t)+\psi'^2(t)}\,dt$$

or

$$\int_L f(x, y)\,ds = \int_a^b f(x, g(x))\sqrt{1+g'^2(x)}\,dt$$

if L is given by the explicit equation $y=g(x)$.

For a 3D-area V, a scalar field $f(x,y,z)$, and a 3D-curve L the line integral is defined in much the same way. If $f(x,y,z)$ is continuous on L given by $x=\varphi(t)$ $y=\psi(t)$ $z=\tau(t)$, $t\in[a,b]$

we can use the formula

$$\int_L f(x,y,z)\,ds=\int_a^b f(\varphi(t),\psi(t),\tau(t))\sqrt{\varphi'^2(t)+\psi'^2(t)+\tau'^2(t)}\,dt$$

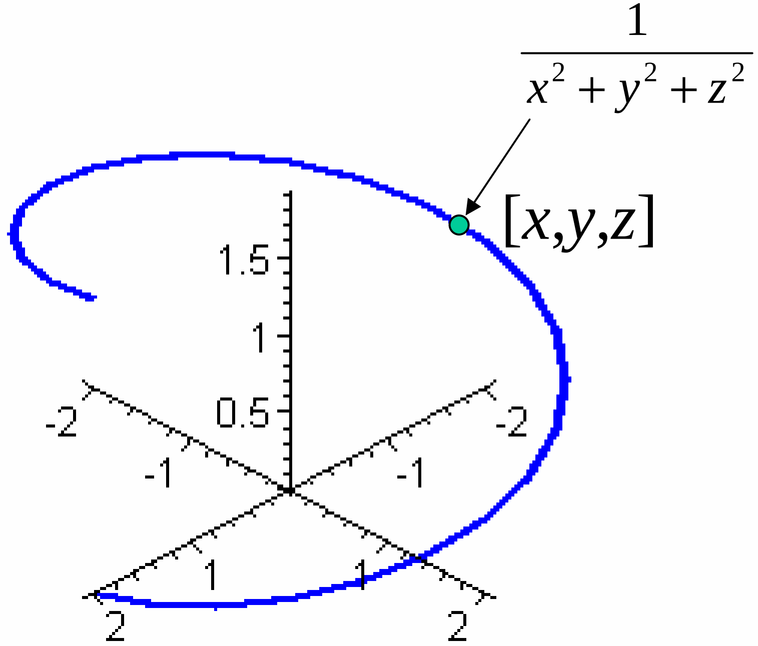
If we recall the way the length of a curve is defined, we see that ds actually denotes the differential of the length of L.

EXAMPLE

Calculate $\int_C \frac{ds}{x^2+y^2+z^2}$ where C is the first turn of the helix

$$x = a\cos t,\ y = a\sin t,\ z = bt, \quad t \in [0, \pi]$$

In the plot below, we have *a*=2 and *b*=0.3



$$\int_0^{2\pi} \frac{ds}{x^2+y^2+z^2} = \int_0^{2\pi} \frac{\sqrt{a^2\sin^2 t + a^2\cos^2 t + b^2}}{a^2(\cos^2 t + \sin^2 t) + b^2 t^2}\, dt =$$

$$= \int_0^{2\pi} \frac{\sqrt{a^2+b^2}}{a^2+b^2t^2}\, dt = \frac{\sqrt{a^2+b^2}}{a^2} \int_0^{2\pi} \frac{dt}{1+\left((bt)^2 \big/ a^2 \right)} = \left\{ \frac{bt}{a} = u \right\} =$$

$$= \frac{a\sqrt{a^2+b^2}}{a^2 b} \int_0^{\frac{2b\pi}{a}} \frac{du}{1+u^2} = \frac{\sqrt{a^2+b^2}}{ab} \left[\arctan u\right]_0^{\frac{2b\pi}{a}} =$$

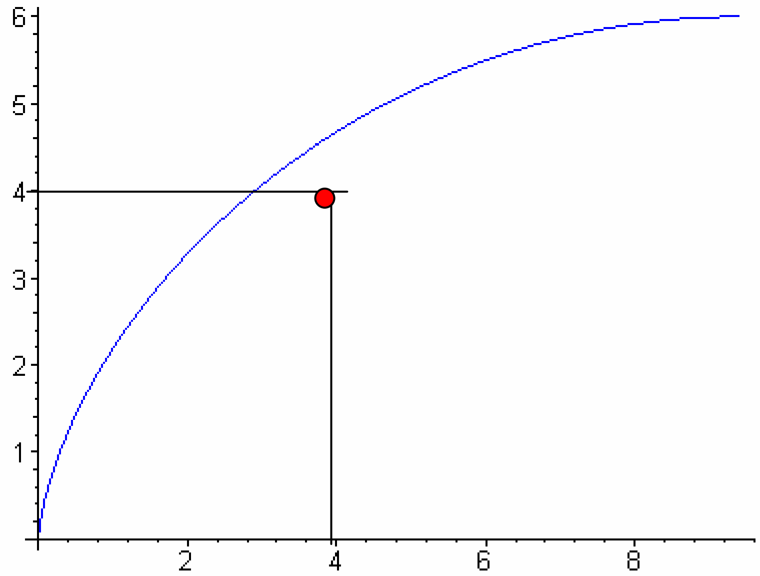
$$= \frac{\sqrt{a^2+b^2}}{ab} \arctan \frac{2b\pi}{a}$$

EXAMPLE

Find the centre of mass of the first half-arch of the cycloid

$$x = a(t - \sin t),\ y = a(1 - \cos t), \quad 0 \le t \le \pi$$

if its specific mass is constant.



The centre of mass of a curve is defined in physics as the point $T=[t_x,t_y]$ that has the following property: if the entire mass of the curve was concentrated at T, its torque with respect to axes x, y would be equal to the torque with respect to axes x, y of all the points added up. This means that

$$t_x = \frac{\int_C x\,ds}{\int_C ds}, \quad t_y = \frac{\int_C y\,ds}{\int_C ds}$$

$$\int_C ds = \int_0^{\pi} \sqrt{(a - a\cos t)^2 + a^2 \sin^2 t}\, dt = \sqrt{2}a \int_0^{\pi} \sqrt{1-\cos t}\, dt =$$

$$= \sqrt{2}a \int_0^{\pi} \frac{2\sqrt{1-\cos t}}{2}\, dt = 2\sqrt{2}a \int_0^{\pi} \sin\frac{t}{2}\, dt = -4\sqrt{2a}\left[\cos\frac{t}{2}\right]_0^{\pi} = 4\sqrt{2}a$$

$$\int_C x\, ds = a^2\sqrt{2} \int_0^{\pi} \sqrt{1-\cos t}\,(t - \sin t)\, dt = a^2\sqrt{2}\left(I_1 - I_2\right)$$

where $I_1 = \int_0^{\pi} t\sqrt{1-\cos t}\, dt$ and $I_2 = \int_0^{\pi} \sin t\sqrt{1-\cos t}\, dt$

$$\int_C y\, ds = a^2\sqrt{2} \int_0^{\pi} \sqrt{1-\cos t}\,(1 - \cos t)\, dt = a^2\sqrt{2}\left(I_3 - I_4\right)$$

where $I_3 = \int_0^{\pi} \sqrt{1-\cos t}\, dt$ and $I_4 = \int_0^{\pi} \cos t\sqrt{1-\cos t}\, dt$

$$I_3=\int_0^{\pi}\sqrt{1-\cos t}\;dt=2\int_0^{\pi}\sin\frac{t}{2}\;dt=4\int_0^{\pi/2}\sin u\;du=4$$

$$I_1=\int_0^{\pi}t\sqrt{1-\cos t}\;dt=\left[-4t\cos\frac{t}{2}\right]_0^{\pi}+4\int_0^{\pi}\cos\frac{t}{2}\;dt=0+8=8$$

$$I_2=\int_0^{\pi}\sin t\sqrt{1-\cos t}\;dt=2\int_0^{\pi}\sin t\sin\frac{t}{2}\;dt=\left\{\frac{t}{2}=u\right\}=$$

$$=4\int_0^{\pi/2}\sin 2u\sin u\;du=8\int_0^{\pi/2}\sin^2 u\cos u\;dt=\{\sin u=v\}=8\int_0^{1}t^2\;dt=\frac{8}{3}$$

$$I_4=\int_0^{\pi}\cos t\sqrt{1-\cos t}\;dt=2\int_0^{\pi}\cos t\sin\frac{t}{2}\;dt=\left\{\frac{t}{2}=u\right\}=$$

$$= 4\int_0^{\pi/2} \cos 2u \sin u \, du = 4\int_0^{\pi/2} \cos^2 u \sin u - \cos^3 u \, dt =$$

$$= 4\int_0^{\pi/2} \cos^2 u \sin u - \cos u + \cos u \sin^2 u \, dt =$$

$$= 4\left[\frac{t^3}{3}\right]_0^1 - 4\left[\sin t\right]_0^{\pi/2} + 4\left[\frac{t^3}{3}\right]_0^1 = \frac{8}{3} - 4 + \frac{8}{3} = \frac{-4}{3}$$

$$\int_C x \, ds = a^2\sqrt{2}\left(I_1 - I_2\right) = \frac{16a^2\sqrt{2}}{3}$$

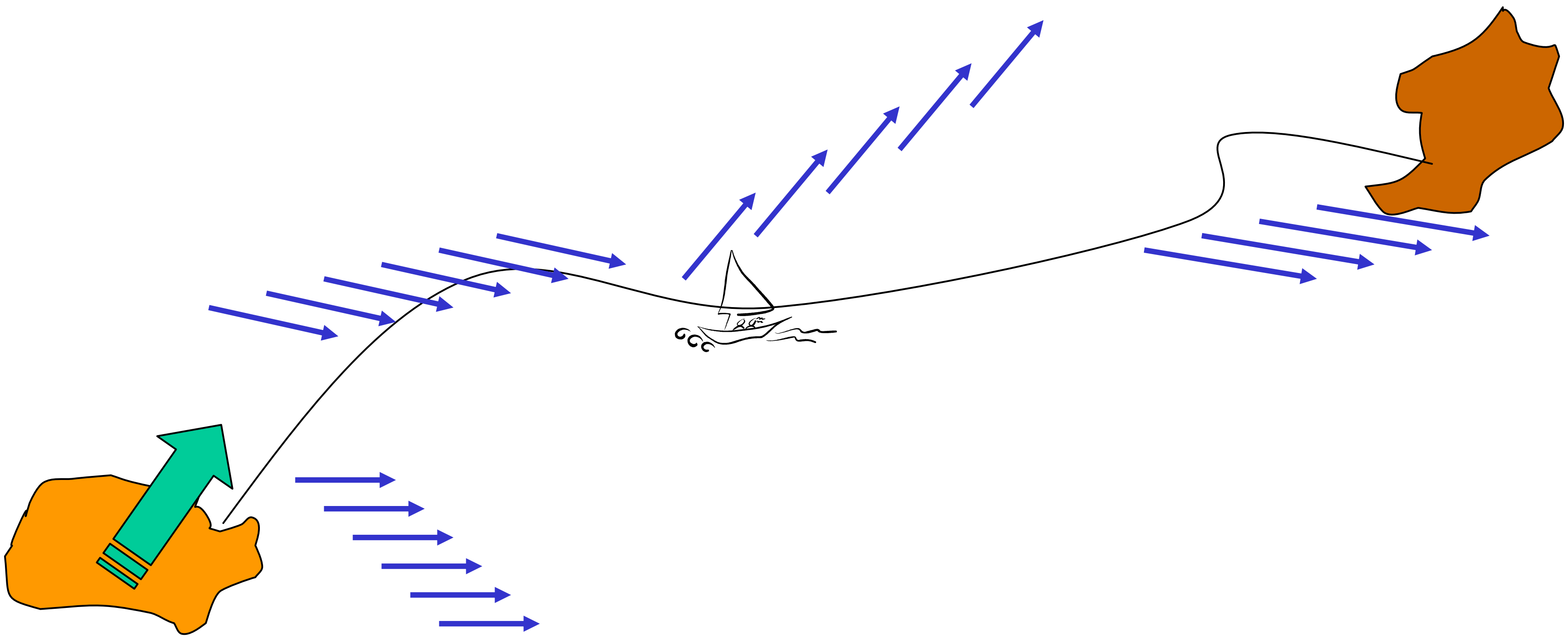
$$\int_C y \, ds = a^2\sqrt{2}\left(I_3 - I_4\right) = \frac{16a^2\sqrt{2}}{3}$$

$$t_x = t_y = \frac{\frac{16a^2\sqrt{2}}{3}}{4\sqrt{2}a} = \frac{4a}{3}$$

LINE INTEGRAL IN A VECTOR FIELD

MOTIVATION

A ship sails from an island to another one along a fixed route.
Knowing all the sea currents, how much fuel will be needed ?



VECTOR FIELD

Let P be a planar contiguous area with a vector function

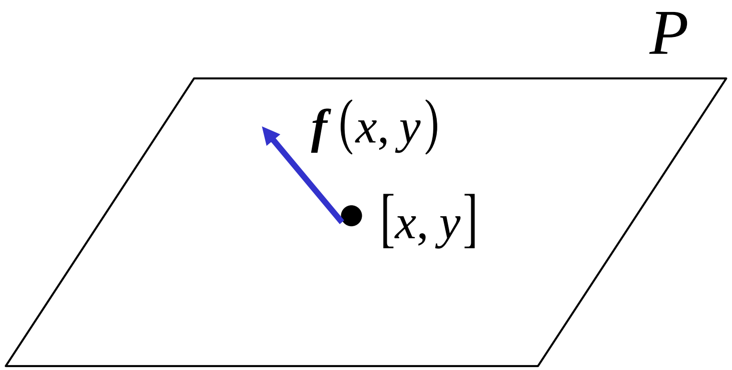
$\vec{f}(x, y) = f_1(x, y)\vec{i} + f_1(x, y)\vec{j}$ defined for each $[x, y] \in P$.

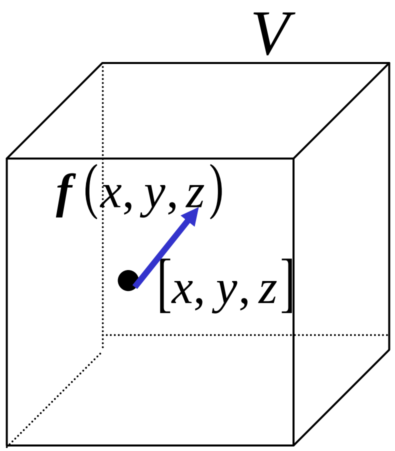
We say that $\vec{f}(x, y)$ is a vector field on P. Similarly,

$\vec{f}(x, y, z) = f_1(x, y, z)\vec{i} + f_2(x, y, z)\vec{j} + f_3(x, y, z)\vec{k}$ is a vector field

in a 3D area V.

water flow, electromagnetic field, ...





Let a vector field $\vec{f}(x, y)$ be defined on a planar area *M*.

Let *L* be an oriented regular curve defined by the parametric equations $x=\varphi(t)$ $y=\psi(t)$, $t\in[a,b]$

Define a partition of *T*:

$$a = a_0 < a_1 < \cdots < a_n$$

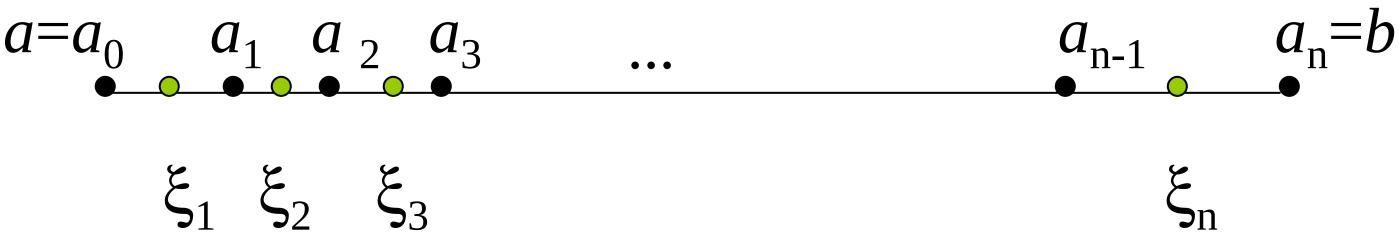
Put $L_0 = [\varphi(a_0), \psi(a_0)], L_1 = [\varphi(a_1), \psi(a_1)], \ldots, L_n = [\varphi(a_n), \psi(a_n)]$

Choose points $\xi_1, \xi_2, \ldots, \xi_n$ such that

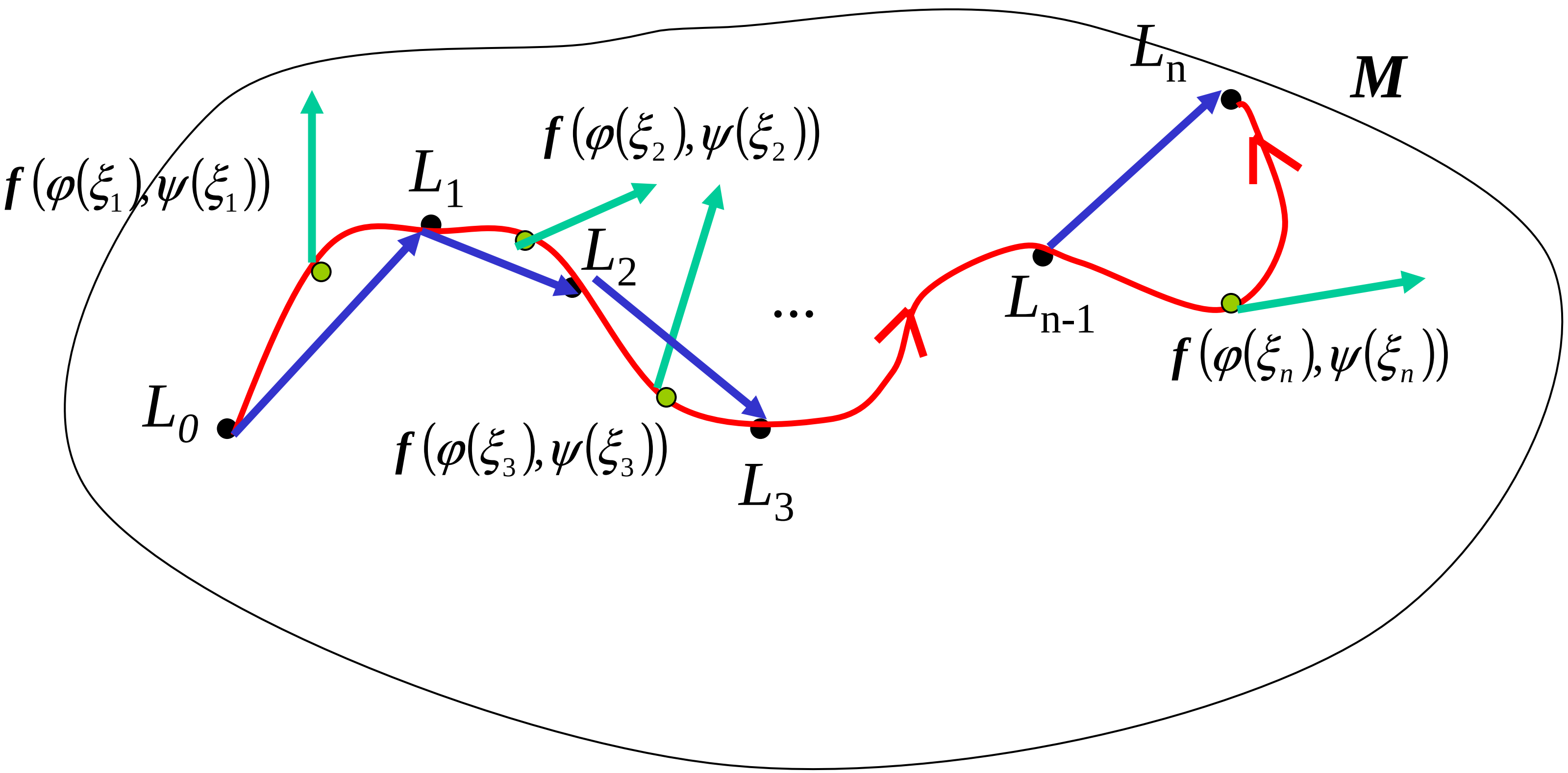
$$a_0 < \psi_1 < a_1 < \psi_2 \cdots < \psi_n < a_n$$

Put $N(a_0, a_1, \ldots, a_n) = \max_{0<i\le n}\left\{\overline{L_i L_{i-1}}\right\}$

$N(a_0, a_1, \ldots, a_n)$ is called the norm of the partition.



$a = a_0$
a_1
a_2
a_3
...
a_{n-1}
$a_n = b$
ξ_1
ξ_2
ξ_3
ξ_n



L_n
M
$f(\varphi(\xi_2), \psi(\xi_2))$
$f(\varphi(\xi_1), \psi(\xi_1))$
L_1
L_2
...
L_{n-1}
$f(\varphi(\xi_n), \psi(\xi_n))$
L_0
$f(\varphi(\xi_3), \psi(\xi_3))$
L_3

Define the following integral sum

$$\mathcal{L} = \sum_{i=1}^{n} \overrightarrow{L_{i-1}L_i}\, \boldsymbol{f}\left(\varphi(\xi_i), \psi(\xi_i)\right)$$

If, for $N(a_0, a_1, \ldots, a_n) \to 0$, we have $\mathcal{L} \to \mathcal{L}_0$ for some finite $\mathcal{L}_0$, we say that $\mathcal{L}_0$ is the ***line*** (or ***curvilinear***) ***integral*** of the vector field $\vec{f}(x, y)$ over L (line integral of the second type).

Symbolically, we write

$$\mathcal{L}_0 = \int_L f(x, y)\, \boldsymbol{ds}$$

or

$$\mathcal{L}_0 = \int_L f_1(x, y)\, dx + f_2(x, y)\, dy$$

$$\mathcal{L}_0 = \int_L f(x,y)\,\boldsymbol{ds} = \int_L f_1(x,y)dx + f_2(x,y)dy$$

is independent of the way we construct the partition of [*a*,*b*] and choose the numbers $\xi_1, \xi_2, \ldots, \xi_n$

To calculate the line integral we can use the formula

$$\int_L f(x,y)\,\boldsymbol{ds} = \varepsilon\int_a^b f_1(\varphi(t),\psi(t))\varphi'(t) + f_2(\varphi(t),\psi(t))\psi'(t)\,dt$$

where ε is 1 if the curve is oriented in correspondence with its parametric equations otherwise it is equal to –1.

For a 3D-area *V*, a vector field $\vec{f}\left(x, y, z\right)$ and a 3D-curve *L* the line integral is defined in much the same way with the following formula for calculating a 3D line integral:

$$\int_L \vec{f}\left(x, y, z\right) \overrightarrow{ds} = \varepsilon \int_a^b \left(f_1\left(\varphi,\psi,\tau\right)\varphi' + f_2\left(\varphi,\psi,\tau\right)\psi' + f_3\left(\varphi,\psi,\tau\right)\tau' \right) dt$$

$$ds$$

Line integral of the second type clearly depends on the way the curve is oriented and $\overrightarrow{ds}$ actually denotes the differential of the tangent vector field along *L*.

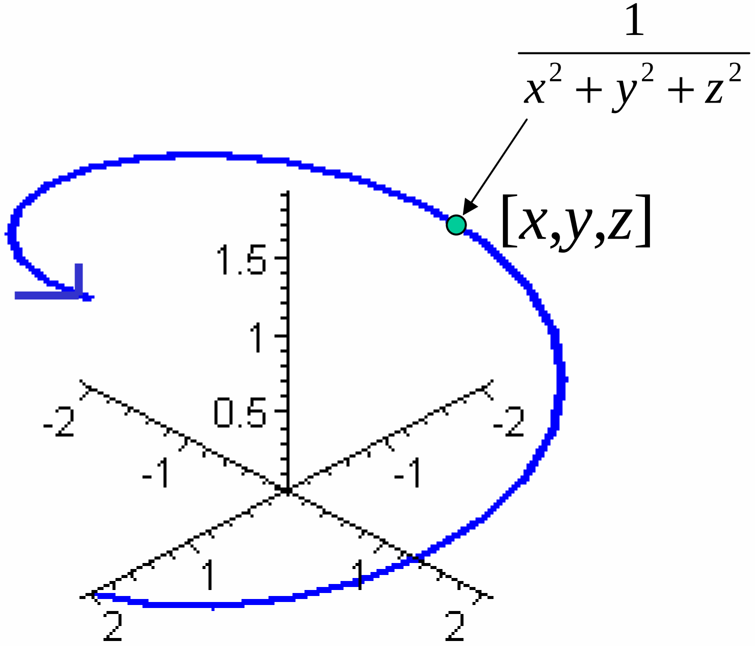
EXAMPLE

Calculate $\int_L (y-z)\,dx + (z-x)\,dy + (x-y)\,dz$

where L is the first turn of the helix oriented in correspondence with the parametric equations

$$x = a\cos t,\ y = a\sin t,\ z = bt, \quad t \in [0, \pi]$$

a=2 and b=0.3



$$\int_L (y-z)\,dx+(z-x)\,dy+(x-y)\,dz=$$

$$=\int_L (a\sin t-bt)(-a\sin t)+(bt-a\cos t)\,a\cos t+(a\cos t-a\sin t)\,b\,dt=$$

$$=\int_0^{2\pi} -a^2\sin^2 t+abt\sin t+abt\cos t-a^2\cos^2 t+ab\cos t-ab\sin t\,dt=$$

$$=ab\left\{\int_0^{2\pi} t\sin t\,dt+\int_0^{2\pi} t\cos t\,dt-\int_0^{2\pi}\cos t\,dt-\int_0^{2\pi}\sin t\,dt\right\}-\int_0^{2\pi} a^2\,dt=$$

$$=ab\left\{\int_0^{2\pi} t\sin t\,dt+\int_0^{2\pi} t\cos t\,dt\right\}-2\pi a^2=$$

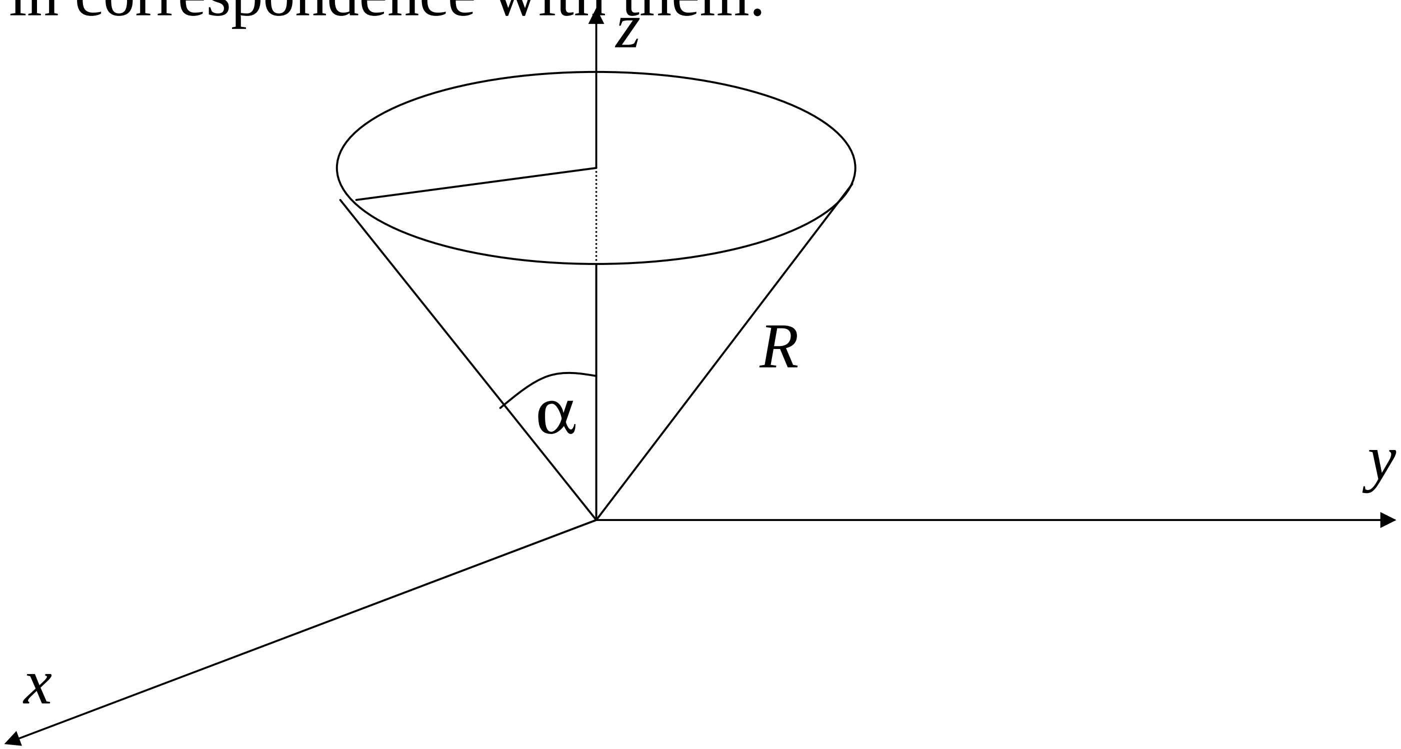
$$=ab\left\{-[t\cos t]_0^{2\pi}+\int_0^{2\pi}\cos t\,dt+[t\sin t]_0^{2\pi}-\int_0^{2\pi}\sin t\,dt\right\}-2\pi a^2=\underline{\underline{-2\pi a(a+b)}}$$

EXAMPLE

Calculate $\oint_C y\,dx + z\,dy + x\,dz$ where C is a circle given by the parametric equations

$$x = R\cos\alpha\cos t,\ y = R\cos\alpha\sin t,\ z = R\sin\alpha,\ \alpha = \text{const}$$

oriented in correspondence with them.



$$\oint_C ydx + zdy + xdz =$$

$$= \int_0^{2\pi} R\cos\alpha \sin t R\cos\alpha(-\sin t) + R\sin\alpha R\cos\alpha \cos t + R\cos\alpha \cos t 0 \, dt =$$

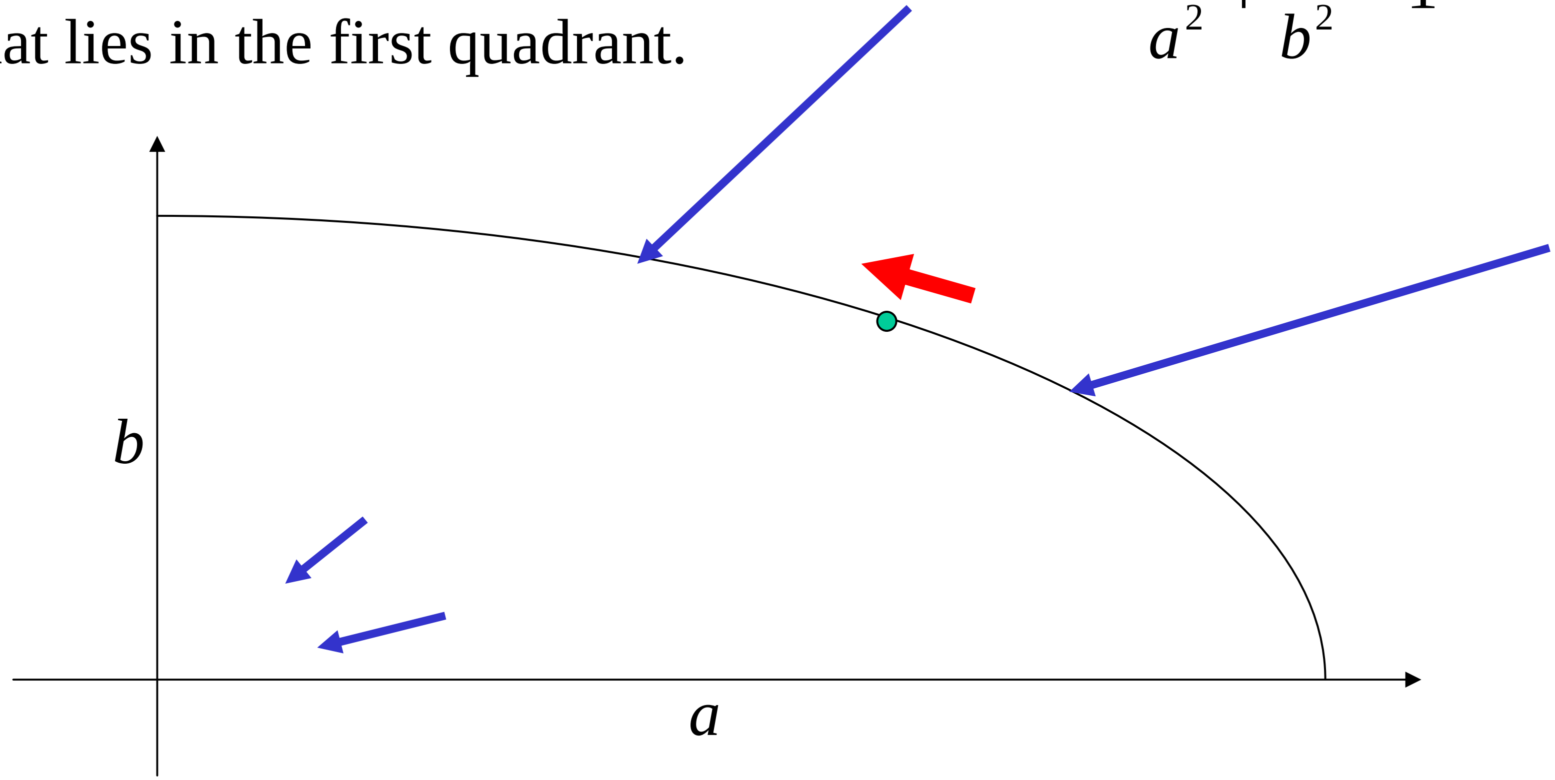
$$= R^2 \int_0^{2\pi} -\cos^2\alpha \sin^2 t + \sin\alpha \cos\alpha \cos t \, dt =$$

$$= -R^2 \cos^2\alpha \int_0^{2\pi} \sin^2 t \, dt + R^2 \sin\alpha \cos\alpha \int_0^{2\pi} \cos t \, dt =$$

$$= -R^2 \cos^2\alpha \int_0^{2\pi} \frac{1-\cos 2t}{2} \, dt = \underline{\underline{-\pi R^2 \cos^2\alpha}}$$

EXAMPLE

Determine the work done by the elasticity force if the vector of this force at each point is directed towards the origin its module being proportionate to the distance from the origin of the point and if the application point of the force moves anticlockwise along the quarter of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ that lies in the first quadrant.



$$W = \int_L f_1(x,y)dx + f_2(x,y)dy \quad \text{where} \quad f_1(x,y) = -kx, \; f_2(x,y) = -ky$$

and L is given by the parametric equations

$$x = a\cos t, \; y = b\sin t, \; t \in \left[0, \pi/2\right]$$

$$W = \int_0^{\pi/2} -ka\cos t(-a\sin t) - kb\sin t b\cos t \, dt =$$

$$= k\left(a^2 - b^2\right)\int_0^{\pi/2} \sin t\cos t \, dt = \{\sin t = u\} =$$

$$= k\left(a^2 - b^2\right)\left[\frac{u^2}{2}\right]_0^1 = \frac{k\left(a^2 - b^2\right)}{2}$$

tends to zero as the ellipse tends to circle