A Computational Model of Groundwater Mound Evolution Using the Complex Variable Boundary Element Method and Generalized Fourier Series

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Abstract. In this work, we propose a numerical scheme for modeling the evolution in time of a groundwater mound on a rectangular domain. The global initial-boundary value problem is assumed to have specified Dirichlet boundary conditions. To model this phenomenon, the global problem is decomposed into two components. Specifically, a steady-state component and a transient component, which are governed by the Laplace and diffusion partial differential equations, respectively. The Complex Variable Boundary Element Method (CVBEM) is used to develop an approximation of the steady-state solution. A linear combination of basis functions that are the product of a two-dimensional Fourier sine series and an exponential function is used to develop an approximation of the transient solution. The global approximation function is the sum of the CVBEM approximation function for the steady-state component and the Fourier series approximation function for the transient part of the global problem. It has been shown in [14] that the global approximation function satisfies the governing partial differential equation.

This work focuses on identifying two key problems for the purpose of assessing the validity of computational groundwater models. The problems considered in this work are important problems in computational geoscience and are related to modeling groundwater mounding. In addition to providing and explaining the two proposed test problems, we also test the performance of the coupled CVBEM and Fourier series partial sum for the purpose of establishing a benchmark standard for other computational models. The methodology for creating test problems that is presented in this work can be applied to creating further such test problems. As more important test problems are developed, more confidence can be obtained regarding the computational veracity of numerical groundwater models.

1. Introduction

The use of computational models for modeling fluid flow is becoming increasingly more useful in problems of various size and complexity. Consequently, an emerging issue in computational engineering mathematics is the general use of computational models to describe groundwater flow. Numerous computational modeling software packages have been developed that are capable of providing approximate solutions to initial-boundary value problems

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that are governed by the fluid flow partial differential equations. Advanced visualization techniques are used in these packages to produce high-quality displays of the problem disposition as well as the results from the computational model. However, although computational groundwater models are useful in understanding and, particularly, visualizing groundwater flow, modeling errors can often result in significant design errors, which can have severe consequences in the construction of important groundwater resource elements such as groundwater recharge basins, for example.

Planning and designing groundwater recharge basins requires an analysis of the unsteady flow of the groundwater that mounds below the recharge basin. Groundwater mounding is a process that can occur when infiltrating stormwater intersects a groundwater table, which typically occurs beneath depressed areas of the Earth such as beneath stormwater infiltration basins (groundwater recharge basins). In these depressed areas, the proximity of surface water to the groundwater table is reduced, and there is a concentration of groundwater recharging [3]. When groundwater recharge is concentrated in a small area, it is possible to cause groundwater mounding [2].

In this paper, the physical process of unsteady flow of groundwater in a groundwater mound is formulated into a set of two test problems that are suitable for modeling with computational methods. These proposed test problems are intended to serve as the beginning of a set of standard problems that can be used in assessing the validity of computational and mathematical techniques for approximating solutions to the unsteady flow problem. The two test problems are mathematical formulations of unsteady groundwater flow and are designed to assess (1) the ability of computational groundwater models to develop descriptions of the potential surface (groundwater surface) and (2) the ability to develop the associated streamline vector trajectories. Other similar types of problems can readily be developed, and the testing procedure can be applied to those alternate problems. Assessments of this type are important for identifying potential flaws in computational packages, but can also be useful for establishing the validity of successful groundwater models, particularly with respect to modeling unsteady groundwater flow.

In order to provide a benchmark for assessing the success of computational groundwater models, we also develop a numerical technique for modeling the evolution in time of a groundwater mound. To model this phenomenon, which is related to unsteady transport, the global problem is decomposed into two components; namely, a steady-state component and a transient component. The steady state component is governed by the Laplace partial differential equation (PDE), $\Delta u_1 = 0$, and the transient component is governed by the diffusion PDE, $\Delta u_2 = \frac{\partial u_2}{\partial t}$. The Complex Variable Boundary Element Method (CVBEM), which is a well-known Laplace solver, is used to develop the approximate potential function description of the steady-state condition. A two-dimensional Fourier sine series is used to develop the approximate potential function description of the transient portion of the groundwater flow equations. The global solution is the sum $u = u_1 + u_2$ of the CVBEM and transient approximation outcomes.

The proposed methodology is applicable to rectangular problem domains as well as problem domains that are the union of several rectangular regions. The boundary conditions of the global initial-boundary value problem are assumed to be Dirichlet. In order to fit the global approximation function to the specified boundary conditions of the global BVP, the boundary conditions of the steady-state component are set to match the boundary conditions of the global problem. Consequently, the boundary conditions of the transient problem are set to be zero continuously (i.e. homogeneous boundary conditions). The homogeneous boundary conditions for the transient problem are the motivation for the use of the two-dimensional Fourier sine series in the basis functions of the transient approximation function. This is because these basis functions can be designed so as to be zero along the boundary of rectangular domains. The initial condition is assumed to be consistent on the boundary of the problem domain with the specified global boundary conditions. It is noted that this methodology can still be applied to model problems with an inconsistent initial condition, however, in that case, the boundary conditions for the transient problem would no longer be homogeneous, which is problematic since the Fourier sine series is identically zero on the boundary of the problem domain.

The CVBEM results in a function that satisfies the Laplace equation within the problem domain. Additionally, it was shown in [14] that the proposed transient solution satisfies the diffusion equation within the problem domain. Thus, the only approximation in this methodology is in approximating the coefficients for the linear combinations of the steady-state and transient approximation functions so as to approximately satisfy the specified boundary and initial conditions, respectively. Therefore, we claim that the numerical solutions to the problems considered in this work that result from the presented methodology can be used as benchmarks by which to assess the accuracy of other computational groundwater models in approximating the solutions to important problems in groundwater flow modeling.

2. Description of Groundwater Test Problems

Many groundwater computational models involve discretizing the problem domain into numerous finite elements, finite volumes, or finite difference grid nodes. Methods to test the modeling veracity of such large groundwater models can be difficult to develop and apply. However, there are several standard problems that computational groundwater models must be able to accurately approximate in order to be confident that they can be used to reliably model more complicated problems. In this work, we propose two such test problems for assessing (1) the ability of computational groundwater models to accurately depict the surface of an evolving groundwater mound, and (2) the ability of computational groundwater models to accurately represent the streamlines of the unsteady groundwater flow. Sufficient performance in modeling these test problems is a necessary, but not sufficient, condition to imply a similar ability to accurately model more complicated problems.

The problems that are proposed were selected because an accurate solution to these problems requires effectively (1) fitting the global approximation function to the initial condition, (2) modeling the unsteady flow (satisfying the governing PDE), (3) fitting the global approximation function to the boundary conditions of the global BVP, (4) generating orthogonal streamlines to the surface potentials. Other test problems that are designed to assess these characteristics could also be useful in assessing the veracity of computational groundwater models.

2.1. Test Problem A: Potential Surface Modeling

An important problem that arises in groundwater management, especially in arid climates, is the banking of surface water in groundwater recharge basins. Such storage causes groundwater mounding, which is a phenomenon that has been well-studied in arid environmental situations. Groundwater mounding is known to cause changes in the surrounding ecosystem such as changes to the vegetative balance and available water supply, among other areas of concern. Additionally, rising groundwater mounds can cause the mobilization and transportation of pollutants that were previously stored in subsurface soil. Therefore, it is important to be able to model and assess the environmental impacts and ramifications of introducing additional water into the soil. Consequently, the accuracy of computational groundwater models is particularly relevant when planning and designing recharge basins.

Test Problem A is designed to assess the accuracy of computational groundwater models in the development of a potential surface that evolves to represent the attenuation of a groundwater mound. The methodology presented in this work is suited for any initial mound condition that satisfies the following two properties: (1) the initial condition is continuous, and (2) the initial condition is consistent on the boundary with the specified global boundary conditions. While many initial conditions could be used, in order to demonstrate the new methodology, a simple, single-peaked mound initial condition is specified. It is assumed that there are no sources of additional groundwater during the modeling time, thus, for model times $t > 0$, the mound continuously reduces in spatial coverage due to downwards and lateral drainage of the stored groundwater in the mound.

The background (i.e., steady-state) groundwater flow regime is assumed to correspond to groundwater flow in a 90-degree bend, which is modeled by the complex variable function $\omega(z) = z^2$. The real and imaginary parts of the function z^2 correspond to the potential and stream function representations of the groundwater flow, respectively. The contours of these two functions, which are depicted in Figure 1, are used to generate the traditional flow net graphical display of the steady-state situation.

The entire test problem is stated formally as:

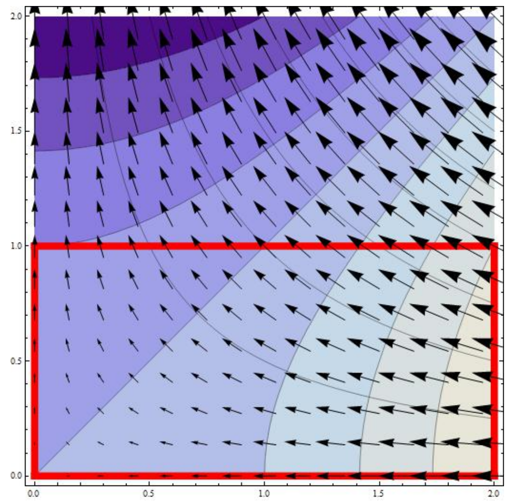
$$\begin{array}{ll} \text{Governing PDE:} & u_{xx} + u_{yy} = u_t \quad \text{on } \Omega = [0,2] \times [0,1] \\ \text{Boundary conditions:} & u(x,y,t) = \Re[z^2] = x^2 - y^2 \quad \text{on } \Gamma \\ \text{Initial condition:} & u(x,y,0) = 100 \sin\left(\dfrac{\pi x}{2}\right) \sin(\pi y) + x^2 - y^2 \end{array}$$

where $\Gamma = \partial\Omega$ is the boundary of the problem domain. The analytic solution of the initial-boundary value problem is

$$u(x,y,t) = 100 \sin\left(\frac{\pi x}{2}\right) \sin(\pi y)\, e^{-\pi^2 (5/4) t} + x^2 - y^2. \tag{1}$$

This flow regime is associated with difficult-to-solve spatial distributions for both the potential and streamline functions, and hence, provides a possibly interesting case where the analysis must predict the dissipation of the flow regimes corresponding to the specified background flow superimposed with the groundwater mound. As the mound dissipates with time, the global flow regime (i.e., background flow plus mound) is restored to just the background flow regime (the steady-state solution). Consequently, the flow streamlines and streamline trajectories are continuously changing in magnitude and direction with model time. For example, in this test problem, flow trajectories actually reverse in many portions of the problem domain with increasing model time.

Figure 1: A contour plot of the steady-state surface corresponding to the flow around a 90-degree bend background regime. The problem boundary is depicted by a red box. The region above the problem domain is depicted for context. The streamlines for the steady-state solution are easily developed by the CVBEM by evaluating the imaginary component of the CVBEM approximation function.



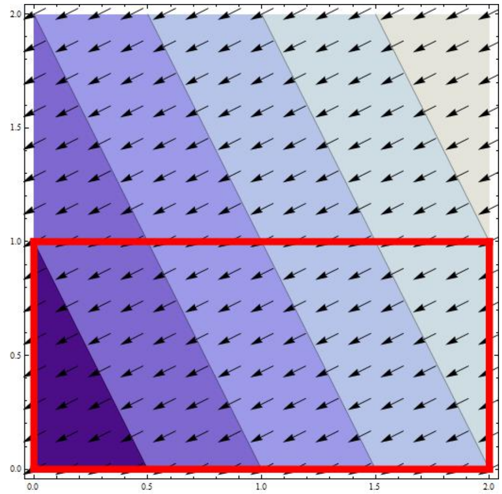
This particular test problem can be applied in conjunction with almost any background flow regime. Using this approach, it is possible to develop a set of suitable test problems that are each potentially useful for testing different aspects of the computational groundwater flow model being examined. Once a set of test problems are developed, the computational groundwater flow model can be applied to each test problem, and its ability to accurately model the phenomenon can be assessed.

2.2. TEST PROBLEM B: Streamline Development

Groundwater flow analysis is often used to assist in the identification of sources of groundwater contamination. Computational models are typically developed and then applied to the flow situation in order to estimate flow streamlines. From the developed approximation of the flow field, it is possible to identify potential locations of sources of groundwater contamination. Therefore, Test Problem B is designed to assess the accuracy of computational groundwater models in the development of streamlines, which indicate the direction of the fluid flow in the groundwater mound. The approach used in Section 2.1 for developing test problems to assess computational groundwater mounding models can also be used in this section to develop test problems for the modeling and identification of groundwater contaminant sources.

The initial condition of this problem is specified as a single-peaked mound. However, it is noted that other initial conditions are possible provided that they conform to the two requirements specified in Section 2.1 regarding the suitability of arbitrary initial conditions. Similarly to the problem in Section 2.1, a groundwater recharge bulge is considered, however, in order to demonstrate another background groundwater flow regime, the steady-state surface models planar flow rather than flow around a 90-degree bend. For this test problem, the plane will be given by $f(x, y) = 2x + y$. Figure 2 displays the behavior of the planar steady-state flow situation and depicts isocontours as well as the vector field.

Figure 2: A contour plot of the steady-state surface corresponding to the planar flow background regime. The problem boundary is depicted by a red box. The region above the problem domain is depicted for context. The streamlines for the steady-state solution are easily developed by the CVBEM by evaluating the imaginary component of the CVBEM approximation function.



The entire test problem is stated formally as:

Governing PDE: $u_{xx} + u_{yy} = u_t$ on $\Omega = [0,2] \times [0,1]$

Boundary conditions: $u(x,y,t) = 2x + y$ on Γ

Initial condition: $u(x,y,0) = 100 \sin\left(\frac{\pi x}{2}\right) \sin(\pi y) + 2x + y$

where $\Gamma = \partial\Omega$ is the boundary of the problem domain.

As the mound dissipates with time, the global flow regime (i.e., background flow plus mound) is restored to just the background flow regime (the steady-state solution). The analytic solution of this initial-boundary value problem is

$$u(x,y,t) = 100 \sin\left(\frac{\pi x}{2}\right) \sin(\pi y)\, e^{-\pi^2 (5/4) t} + 2x + y. \tag{2}$$

3. Numerical Method Development

3.1. General Groundwater Transport Formulation

The general partial differential equation governing groundwater flow is provided in Equation (3),

$$C\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(K_x \frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(K_y \frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial z}\left(K_z \frac{\partial u}{\partial z}\right) + \sum_{k=1}^{n} S_k \tag{3}$$

where, C is a capacitance coefficient; x, y, and z are the spatial coordinates; t is the model time coordinate; ϕ is the potential function; K_i for $i = x, y, z$ is the hydraulic conductivity, with each subscript denoting the specific coordinate direction; and S_k is a source or a sink.

For further details regarding the development of these flow equations, the reader is referred to [16, 1].

In this work, we consider a problem in two spatial dimensions; namely, x and y. Additionally, the problem domain is assumed to be homogeneous and isotropic, which reduces the need for parameter specification in the problem formulation. So, we assume the following constant values $C = K_x = K_y = 1$ and, we also assume that there are no sources or sinks. Under these assumptions, Equation (3) reduces to

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \Delta u,$$

which is the diffusion partial differential equation.

The flow situation considered in this work involves modeling the unsteady flow of a groundwater mound, which is resolving towards the steady-state conditions of the problem as indicated by the specified global boundary conditions. Therefore, the conceptual problem can be decomposed into two components, (1) an unsteady flow component of the groundwater mound, which is reducing in vertical extent over time due to groundwater flowing into the underlying (background) groundwater regime and (2) a steady-state component representing the considered test situation after the groundwater mound has fully drained into the background flow regime. The global approximation function is the sum $u = u_1 + u_2$ of the approximation functions for the steady-state and transient components.

The unsteady flow component (or transient component) is modeled by the product of a two-dimensional Fourier sine series in the spatial variables x and y and an exponential function in the model time variable t. Since the geometry is rectangular, it is tractable to the development of a two-dimensional Fourier sine series approximation of the governing unsteady flow equations for homogeneous boundary conditions. The steady-state component is modeled by the Complex Variable Boundary Element Method procedure using complex variable monomials as basis functions.

In order to fit the global approximation function to the specified boundary conditions of the global BVP, the boundary conditions of the steady-state component are set to match the boundary conditions of the global problem. Consequently, the boundary conditions of the transient problem are set to be zero continuously (i.e. homogeneous boundary conditions). The homogeneous boundary conditions for the transient problem are the motivation for the use of the two-dimensional Fourier sine series in the basis functions of the transient approximation function. The initial condition for the transient component of the global problem is specified as the difference between the global initial condition and the CVBEM approximation of the steady-state solution.

3.2. CVBEM Approximation of the Steady-State Component

The governing PDE for the steady-state component of the global problem is the Laplace equation $\Delta u_1 = 0$. The CVBEM, which is a well-known Laplace solver, is a linear combination of analytic complex variable basis functions of the form

$$\hat{\omega}(z) = \sum_{k=1}^{p} c_k g_k(z), \tag{4}$$

where c_k is the k^{th} complex coefficient, $g_k(z)$ is the k^{th} member of the family of basis functions being used in the approximation, and p is the number of basis functions being used in the approximation. The CVBEM is the topic of numerous papers and books including [15, 12, 13, 4, 6, 11, 5, 10], and it is assumed that the reader is familiar with the mechanics of the CVBEM. Therefore, many details of this solution technique are not discussed here. However, if more information regarding the theoretical development of the CVBEM is desired, the reader is referred to [8], [9], and [13].

To approximate a solution to the steady-state problem, the CVBEM is applied to the boundary conditions of the global BVP. Complex monomials are used in the current work as the family of basis functions in the CVBEM formulation, however, any family (or combination of families) of analytic basis functions could be used. Since analytic complex variable basis functions are used in the CVBEM development, both the real and the imaginary component functions of the CVBEM approximation function satisfy the Laplace equation. Collocation of the CVBEM approximation function with the specified global boundary conditions is used to determine the coefficients of the linear combination of the CVBEM approximation function. Since the coefficients of the CVBEM linear combination are imaginary, each coefficient results in two degrees of freedom to be determined during the CVBEM modeling process. Thus, in order to uniquely determine the coefficients of the CVBEM approximation function, it is necessary to specify boundary conditions at two unique locations for each term in the CVBEM approximation function.

Determining which part of the CVBEM approximation function to use during collocation depends on the type of boundary conditions that are specified in the global boundary value problem. If boundary conditions from the potential function (Dirichlet) are specified, then collocation would proceed with the real component of the basis functions. Likewise, if given boundary conditions from the stream function (Neumann), those boundary conditions would be collocated with the imaginary component of the basis functions. Mixed boundary value problems are modeled by collocating the real and imaginary component basis functions with the potential and stream boundary conditions, respectively.

Once the coefficients are known, it is possible to approximate the potential function of the steady-state situation by applying the coefficients to the real part of the CVBEM approximation function. Likewise, it is possible to approximate the corresponding stream function by applying the coefficients to the imaginary part of the CVBEM approximation function. Note that it is possible to approximate all of the streamline values within the problem domain without knowing any streamline boundary conditions. That is, the equation for the stream function is a direct product of the CVBEM due to the orthogonality of the real and imaginary components of the CVBEM approximation function. Accomplishing this with real variable domain techniques such as the Finite Element Method would require post-processing involving an additional numerical scheme to approximate the orthogonal streamlines.

3.3. Fourier Series Approximation of the Transient Component

The transient component of the global initial-boundary value problem is governed by the PDE $\Delta u_2 = \frac{\partial u_2}{\partial t}$. The boundary conditions of this problem are specified to be continuously zero, which motivates the use of a two-dimensional Fourier sine series, which can be designed so as to be continuously zero on the boundary of a rectangular problem domain. The initial condition of this problem is specified as the difference between the initial condition of the global initial-boundary value problem and the CVBEM approximation of the steady-state solution. The approximate transient solution, which is given below and denoted $\hat{u}_2$, is a linear combination of basis functions that are the products of a two-dimensional Fourier Sine Series and an exponential function.

$$\hat{u}_2(x,y,t) = \sum_{i=1}^{m}\sum_{j=1}^{n} a_{i,j}\sin\left(\frac{\pi x i}{L_1}\right)\sin\left(\frac{\pi y j}{L_2}\right)e^{-\pi^2\left(\frac{i^2}{L_1^2}+\frac{j^2}{L_2^2}\right)t} \quad (5)$$

where $\hat{u}_2$ is the approximate value of the potential quantity that is associated with the unsteady component of the problem at a particular location and time, x and y are spatial variables, t is the model time, i and j are indices, $a_{i,j}$ is the coefficient corresponding to the $(i,j)^{\text{th}}$ term of the series (to be determined by collocation with the given initial condition), and L_1 and L_2 are the length and width of the rectangular domain, respectively.

One collocation point is needed for each term of the series in Equation (5). Therefore, it is necessary to specify the initial condition at mn distinct points within the problem domain. In general, these initial condition collocation points should be located reasonably uniformly spaced throughout the problem domain.

Notice that since sine functions are used it the Fourier series, which are zero whenever $x=0$, $x=L_1$, $y=0$, or $y=L_2$, Equation (5) is zero continuously along the boundary of the rectangular problem domain. Therefore, the boundary conditions of the transient problem are satisfied inherently by the transient approximation function. In order to fit the initial condition, we consider the function in Equation (5) when evaluated at $t=0$. This is

$$\hat{u}_2(x,y,0) = \sum_{i=1}^{m}\sum_{j=1}^{n} a_{i,j}\sin\left(\frac{\pi x i}{L_1}\right)\sin\left(\frac{\pi y j}{L_2}\right) \quad (6)$$

The coefficients $a_{i,j}$ in Equation (6) are determined by collocation of the truncated Fourier series approximate solution with the initial condition of the transient problem. It is necessary to specify mn distinct domain collocation points in order to uniquely determine the coefficients of the approximation function for the transient component. Once the coefficients are known, they can be substituted back into Equation (5) and can then be used to approximate all of the potential values corresponding to the transient component within the problem domain. This is the approximation of the transient solution.

4. Numerical Solutions to Test Problems Using Coupled CVBEM and Fourier Series Approximation

Groundwater flow vector gradients are determined as standard vector gradients of the resulting global potential function outcome. Since both the CVBEM outcome as well as the Fourier series approximation of the transient solution are functions, it is possible to calculate the gradient of their sum, which represents the global approximation function. This results in a vector field representing streamlines, which are orthogonal to the iso-potential lines.

The global approximation functions that were used in assessing the maximum error of the global approximation function for various time steps was created using eight terms in the

CVBEM approximation function and eight terms in the transient solution approximation function. The maximum errors that are presented in this section were approximated by comparison of the global approximation with the analytic solution at 2,500 uniformly spaced points within the problem domain.

4.1. Approximation of Test Problem A: Potential Surface Modeling

Figure 3 shows the two-dimensional flow field vector trajectories corresponding to the dissipation of the mound in Test Problem A for several model time instances. From these vector plots, it is seen that as the groundwater mound dissipates with time, the flow regime vector field transforms from a combined flow field regime into the flow field representing groundwater flow in a 90-degree bend (the steady-state solution).

Table 1: Maximum relative error in the approximation of the initial condition

Model time	Maximum absolute error	Model time	Maximum absolute error
0	4.2632e−14	0.6	1.7763e−15
0.1	1.0658e−14	0.7	1.7763e−15
0.2	5.3290e−15	0.8	1.7763e−15
0.3	1.7763e−15	0.9	1.7763e−15
0.4	1.7763e−15	1.0	1.7763e−15
0.5	1.7763e−15	Steady-state	1.7763e−15

Notice that as model time increases, the maximum absolute error approaches the maximum absolute error of the steady-state solution.

4.2. Approximation of Test Problem B: Streamline Development

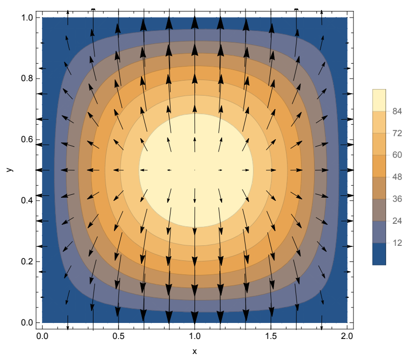
Figure 4 shows the two-dimensional flow field vector trajectories corresponding to the dissipation of the mound in Test Problem B for several model time instances. From these vector plots, it is seen that as the groundwater mound dissipates with time, the flow regime vector field transforms from a combined flow field regime into the flow field representing planar flow (the steady-state solution).

Table 2: Maximum relative error in the approximation of the initial condition

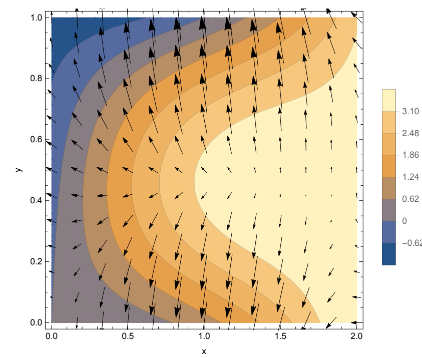
Model time	Maximum absolute error	Model time	Maximum absolute error
0	4.2632e−14	0.6	2.6645e−15
0.1	7.1054e−15	0.7	2.6645e−15
0.2	3.5527e−15	0.8	2.6645e−15
0.3	2.6645e−15	0.9	2.6645e−15
0.4	2.6645e−15	1.0	2.6645e−15
0.5	2.6645e−15	Steady-state	2.6645e−15

Notice that as model time increases, the maximum absolute error approaches the maximum absolute error of the steady-state solution.

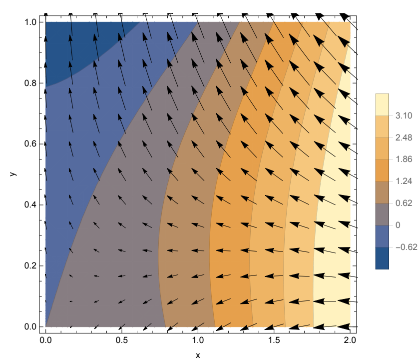
Figure 3: Time evolution of groundwater mound with underlying flow around a 90-degree bend.



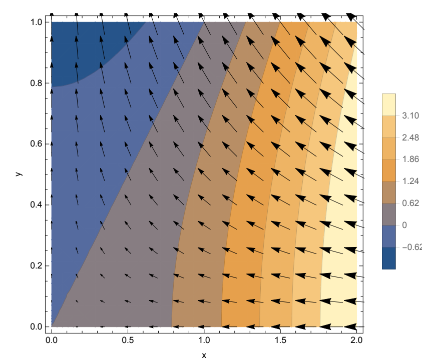
(a) Mound evolution at model time $t = 0.0$.



(b) Mound evolution at model time $t = 0.3$.



(c) Mound evolution at model time $t = 0.5$.



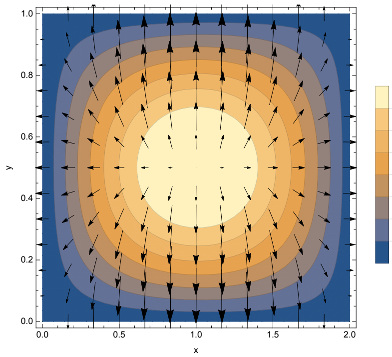
(d) Mound evolution at model time $t = 1.0$.

5. Conclusions and Discussions

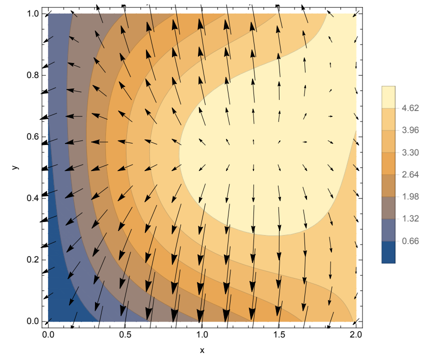
Increasingly, the use of computational models for approximately solving problems in fluid flow, including groundwater flow, has resulted in modeling with large-scale computational models involving millions of computational elements. Numerous computational modeling computer packages are available that provide approximate solutions to the governing groundwater flow equations and account for problem-specific boundary and initial conditions. The software packages are frequently capable of providing various high-quality displays including depictions of the problem setting as well as depictions of the computational results. However, the high-quality visual displays sometimes provide users with a false sense of confidence in the computational outputs of the model. Thus, as these visual displays improve, and as the use of commercially- and publicly-available modeling software becomes more common, it is becoming increasingly more important to develop a procedure for validating the modeling results computationally.

In this work, an important problem in the planning and design of groundwater management systems is considered; namely, the unsteady flow description of a groundwater mound located below a groundwater recharge basin. In this paper, the physical process is formulated as a test problem that is suitable for use in assessing computational groundwater flow models. We propose assessing the outcomes generated by computational softwares with respect to both their approximations of the potential surface (groundwater surface) as well as their approximations of the streamline vector trajectories. Two test problems are developed

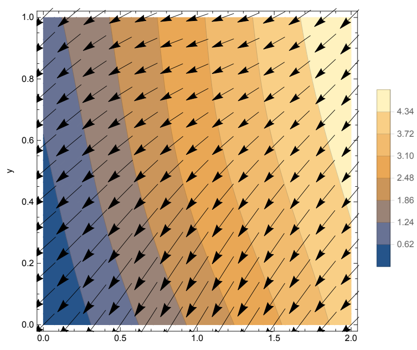
Figure 4: Time evolution of groundwater mound with underlying planar flow regime.



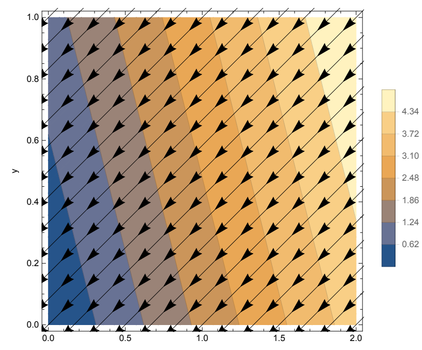
(a) Mound evolution at model time $t = 0.0$.



(b) Mound evolution at model time $t = 0.3$.



(c) Mound evolution at model time $t = 0.5$.



(d) Mound evolution at model time $t = 1.0$.

and considered in this paper, and other similar types of problems can be readily developed and employed by following the testing procedure set forth in this work. Such an assessment may lead to better confidence in computational results and possibly an increased ability to identify potential computational modeling issues with respect to modeling groundwater flow.

Further, in order to provide a benchmark standard for the purpose of assessing other computational models, we also developed a numerical scheme for modeling unsteady groundwater flow problems on rectangular domains with Dirichlet boundary conditions and a consistent initial condition. The modeling procedure is based on the standard approach of resolving the global initial-boundary value problem into a steady-state component and a transient component. The PDE governing the steady-state component is the Laplace equation, and the PDE governing the transient component is the diffusion equation. The boundary conditions for the steady-state component are set to match the boundary conditions of the global BVP. The boundary conditions for the transient component are prescribed to be zero continuously, and the initial condition is specified as the difference between the global initial condition and the CVBEM approximation of the steady-state solution. The coefficients for the CVBEM approximation function as well as the transient approximation function are determined by collocation. The global approximation function is the sum of the CVBEM approximation function and the transient approximation function. The global approximation function satisfies the governing PDE.

This procedure can be extended to three-dimensional problems using a three-dimensional

Fourier sine series for the transient portion of the global problem, and the three-dimensional CVBEM approach to solve the boundary value problem corresponding to the steady-state portion of the global problem [7].

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